A COMPLEXITY DICHOTOMY FOR SATISFIABILITY OVER PARTIAL ORDERS

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INTRODUCTION

Reasoning about temporal data is a common task in various areas of computer science, including Artificial Intelligence, Computational Linguistics and Operations Research. A typical computational problem in this context is Scheduling: Given a set of events and a set of temporal constraints on them, is there a time assignment of the events that satisfies all the constraints?

Usually time in such problems is modeled by linear orders. But for instance to model distributed and parallel computing, partial orders are more suitable, see Lamport [3]. We classify the computational complexity of all scheduling problems, where time is modeled by a partial order. Depending on the type of constraints, we show that such a problem is always either in P or NP-complete [4].

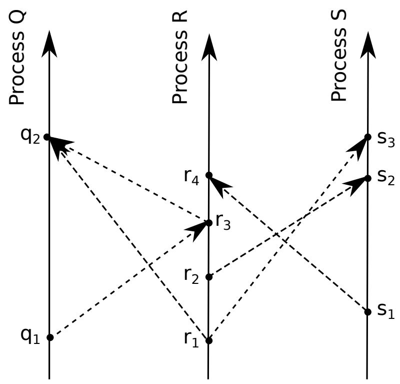

Figure 1: An example of parallel processing modeled as partial order; arrows denote the order/precedence relation

THE PROBLEMS

Let Φ be a finite set of quantifier-free formulas over the relational symbol $\leq$. The formulas in Φ determine the types of constraints that we allow as input. Then we define Poset-SAT(Φ) as the following computational problem:

Poset-SAT(Φ):
INPUT: Variables $\{x_1,\ldots,x_n\}$ and a set of formulas $\phi_i(x_{i_1},\ldots,x_{i_k})$, where $\phi_i \in \Phi$;
QUESTION: Is there a partial order $(A;\leq)$ such that the set of formulas is satisfied in A, i.e., are there $a_1,\ldots,a_n \in A$ such that $\phi_i(a_{i_1},\ldots,a_{i_k})$ holds in A for all ϕ_i?

Our aim is to give a complete classification of the complexity of such problems. It is easy to see that Poset-SAT(Φ) can be always solved in nondeterministic polynomial time. We show that an analogue to Schafer's famous dichotomy for Boolean satisfiability problems [5] holds: Poset-SAT(Φ) can be solved either in P or is NP-complete, depending on the allowed constraints Φ.

PROOF STRATEGY

In proving our result, the main problem is not to identify the complexity for single instances of Φ, but to determine *all* possible sources of NP-completeness or tractability. To do so we use a variety of methods from complexity theory, universal algebra and model theory that were developed by Bodirsky and Pinsker to tackle the analogous problem for graphs [2].

Every Poset-SAT problem can be restated using the *random partial order* $(P;\leq)$, a well-known structure in model theory that can be defined as the unique countable partial order that is homogeneous and contains an isomorphic copy of every finite partial order. By this universality property, the question if there is a solution to an input formula of Poset-SAT(Φ) is equivalent to the question if there are elements of $(P;\leq)$ satisfying this formula.

Hence our strategy is to investigate the random partial order $(P;\leq)$ respectively the structure P_Φ defined on P by the formulas in Φ. The nice model-theoretical properties of the random partial order allow us to do so by studying the *polymorphism clone* of P_Φ instead, i.e. the set of all functions from P^n to P that preserve all relations of P_Φ; in fact the complexity of Poset-SAT(Φ) only depends on this polymorphism clone. Hence the classification of Poset-SAT(Φ) problems translates directly to the investigation of polymorphism clones of structures P_Φ; the smaller the clone, the harder the induced problem. The so called *method of canonical functions* that relies on Ramsey theoretic properties of $(P;\leq)$ helps us to perform this analysis and draw the line between small NP-complete clones and big clones of tractable structures.

RESULTS AND DISCUSSION

Every problem of the form Poset-SAT(Φ) can be either solved in polynomial time or is NP-complete. We can further precisely describe the border between problems in P and NP-c: Either a relation from a given finite list (Low, Betw, Cycl, Sept...) is primitive positive definable in P_Φ and the problem is NP-complete, or the problem is in P.

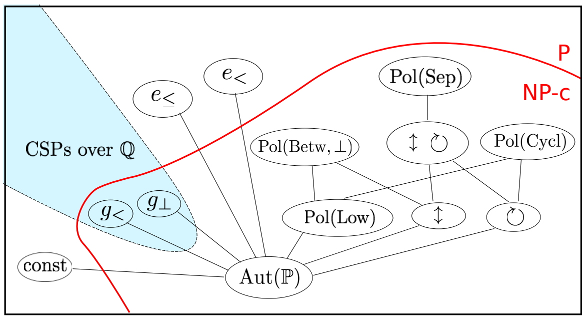

Figure 2: Dichotomy on polymorphism clones

This dichotomy corresponds to an algebraic dichotomy: Either the polymorphism clone of P_Φ contains a function that satisfies some non-trivial equation, and Poset-SAT(Φ) is in P, or the problem is NP-complete [4].

CONCLUSION

In studying constraint satisfaction problems over infinite structures, on one hand we can make statements about computation problems that appear naturally, in this case scheduling problems in distributed computing. On the other hand, the methods that we use to prove such results give us new deep insights about model-theoretic and algebraic properties of the underlying structures, in this case the random partial order and the structures which are first-order definable on it. This exchange between basic research and applications forms new connections between complexity theory, universal algebra and Ramsey theory and is subject of current and ongoing research [1].

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